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2008 J. Phys. A: Math. Theor. 41 285201

(http://iopscience.iop.org/1751-8121/41/28/285201)

J. Phys. A: Math. Theor. **41** (2008) 285201 (14pp) doi:10.1088/1751-8113/41/28/285201

Darboux invariants of integrable equations with variable spectral parameters

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Received 23 April 2008
Published 17 June 2008
Online at stacks.iop.org/JPhysA/41/285201

Abstract
The Darboux transformation for integrable equations with variable spectral parameters is introduced. Darboux invariant quantities are calculated, which are used in constructing the Lax pair of integrable equations. This approach serves as a systematic method for constructing inhomogeneous integrable equations and their soliton solutions. The structure functions of variable spectral parameters determine the integrability and nonlinear coupling terms. Three cases of integrable equations are treated as examples of this approach.

PACS numbers: 42.65.Tg, 05.45.Yv

1. Introduction

Recently, much interest has arisen in the construction of inhomogeneous integrable equations [1–6]. This is because there exist many realistic physics problems in the inhomogeneous systems. These include the inhomogeneous nonlinear Schrödinger equation (NLSE) with variable spectral parameters, which describes the transmission of solitons through the varying dispersion-managed optical fiber [7, 8]. There have also been interesting studies of inhomogeneous NLSEs, including the dc–ac system, the Painlevé test of inhomogeneous NLSEs and construction of modified NLSEs [9–12]. Another development of inhomogeneous integrable equations was the problem of the Heisenberg spin chain having a site-dependent interaction term [13–15], which can be treated by the inverse scattering problem with variable spectral parameters. It has also been shown that the inhomogeneous self-induced-transparency problem, especially with a pumping term, can be described by the complex sine-Gordon equation with variable spectral parameters [11, 16, 17]. All these developments show that a systematic method in the construction of *inhomogeneous integrable equations with variable spectral parameters* is strongly required.

There exists a systematic method in constructing nonlinear integrable equations, especially effective for equations of complex nature, based on the principle of Darboux covariance

[18–21]. The method first finds the *Darboux invariants* of the Lax pair, U and V, and these Darboux invariants are used to determine some matrix elements of U and V. (It has been set to be constants.) Note that the Darboux transformation is used to obtain a new solution for the field variables from an old one. Thus, matrix elements of U and V which are not Darboux invariants are field variables or their functionals. Hayashi and Nozaki [18] were the first to use these Darboux invariants in constructing a Lax pair of integrable equations, $U, V \in \mathrm{su}(2)$. A more general type of integrable equation based on the $U, V \in \mathrm{gl}(2, \mathrm{C})$ was constructed in [19], while [20] used the Darboux invariance to construct AZKN-type integrable equations. A generalization of the method that sets the Darboux invariants to be nonconstants (in fact, explicit functions of z and $\bar{z}$, which will be called *inhomogeneity functions* in the following) have been studied in [22, 23]. This generalization results in a new class of inhomogeneous equations, which were not obtained from the conventional approach. Up to now, all these developments of Darboux invariants method were applied to equations with constant spectral parameters. The equations with nonconstant Darboux invariants and with variable spectral parameters should expand the possible class of inhomogeneous integrable equations. This is the main motivation of the present work.

The Lax pair U, V arises from the associated linear problem of integrable equations,

$$\begin{aligned} \partial\Phi &= \sum_{m=-\bar{M}}^{M} \lambda^m U_m \Phi \equiv U(\lambda)\Phi, \\ \bar{\partial}\Phi &= \sum_{n=-\bar{N}}^{N} \lambda^n V_n \Phi \equiv V(\lambda)\Phi, \end{aligned} \tag{1}$$

where $\partial \equiv \partial/\partial z$, $\bar{\partial} \equiv \partial/\partial\bar{z}$ and λ is the spectral parameter. Here we use the Darboux invariants to construct $U, V \in \mathrm{su}(2)$ for three cases; case A: $(M = 1, \bar{M} = 0)$, case B: $(M = 2, \bar{M} = 0)$ and case C: $(M = \bar{M} = 1)$ for equations with variable spectral parameters, see [18] for cases of constant spectral parameter. For this, the Darboux transformation for equations with variable spectral parameters is introduced and Darboux invariants according to this transformation are constructed (section 2). Some explicit integrable equations are constructed using these Darboux invariant quantities as a guiding principle in constructing U and V (section 3). It is found that the variable spectral parameter, satisfying equation (2), is related to the integrability by requiring $\alpha_M = 0$, as well as the appearance of nonlinear terms with integrals on field variables (through coupling with β_N or β_{N-1}). So the structure of inhomogeneous integrable equations are determined by α_i, β_i of equation (2) which we will call *structure functions* of variable spectral parameters. Section 4 calculates one-soliton solutions for the three cases of integrable equations by using the Darboux transformation.

2. Darboux invariants

2.1. Variable spectral parameter

In this paper, we consider nonlinear integrable equations defined by the Lax pair that can be expressed in terms of 2×2 $\mathbf{su(2)}$ matrices $U_i(\psi, z, \bar{z})$, $V_i(\psi, z, \bar{z})$ in equation (1). Here $\psi(z, \bar{z})$ denotes the field variable of integrable equations (additionally ϕ, R, Q for cases of multi-field variables). U and V for inhomogeneous equations contain *inhomogeneity functions* of $z, \bar{z}$ as well as $\lambda(z, \bar{z})$, $\psi(z, \bar{z})$. The variable spectral parameter $\lambda(z, \bar{z})$ satisfies certain relations of the type, and we consider the following case in this paper:

$$\partial\lambda(z,\bar{z}) = \sum_{m=-\bar{M}}^{M} \lambda^m \alpha_m, \qquad \bar{\partial}\lambda(z,\bar{z}) = \sum_{n=-\bar{N}}^{N} \lambda^n \beta_n, \tag{2}$$

where the structure functions α_i, β_i are real functions of z and $\bar{z}$.

The existence of $\lambda(z,\bar{z})$, i.e., $\partial\bar{\partial}\lambda(z,\bar{z}) = \bar{\partial}\partial\lambda(z,\bar{z})$, constrains the form of α_i and β_i. The properties of equation (2), including the possible forms of α_i and β_i, are discussed in [16]. We will mention their results where necessary.

2.2. *Compatibility of the Lax pair*

The compatibility of two equations in equation (1), i.e., the Lax pair, for any value of $\lambda(z,\bar{z})$ requires

$$\bar{\partial}U_k - \partial V_k + \sum_{m=-\infty}^{\infty}([U_m, V_{k-m}] + m\beta_{k+1-m}U_m - m\alpha_{k+1-m}V_m) = 0. \tag{3}$$

Here we have extended our notation so that $U_m = \alpha_m = 0$, for $m < -\bar{M}$ or $m > M$, and $V_n = \beta_n = 0$ for $n < -\bar{N}$ or $n > N$. Equation (3) gives constraints on $U_m, V_m, \alpha_m, \beta_m$ for $k \neq 0$ (and $k \neq 1$ for case (C)) while it becomes the equation of motion for $k = 0$ (and $k = 1$ for case (C)).

2.3. *Darboux transformation*

Now consider the Darboux transformation [24, 25]

$$\Phi \to \Phi^{[N]} : \Phi^{[N]} = S(\lambda,\lambda_1)[\lambda - \lambda_1^* - (\lambda_1 - \lambda_1^*)P]\Phi \equiv S(\lambda,\lambda_1)[\lambda - \sigma]\Phi. \tag{4}$$

Here the projection operator ($P^2 = P$) is defined

$$P = \frac{\Phi_1\Phi_1^\dagger}{\Phi_1^\dagger\Phi_1}, \tag{5}$$

where the 2-component column matrix Φ_1 is a solution of the linear equation in equation (1) at a specific value of $\lambda = \lambda_1$. The Darboux transformation in the form of equation (4) is introduced in [25] without the $S(\lambda.\lambda_1)$ factor. Here, for equations with variable spectral parameter $S(\lambda.\lambda_1)$ is introduced to make $\det\Phi = \det\Phi^{[N]} = 1$, which is required because $U(\lambda) \in \mathbf{su(2)}$-algebra and $\Phi \in SU(2)$-group in equation (1). A similar form of the Darboux transformation was introduced in [26]. Thus,

$$S(\lambda,\lambda_1) = (\det[\lambda - \sigma])^{-1/2} = \{\lambda^2 - (\lambda_1 + \lambda_1^*)\lambda + \lambda_1\lambda_1^*\}^{-1/2}. \tag{6}$$

By using equation (2), we obtain

$$S^{-1}\partial S = -\sum_{n=1}^{n=M}\alpha_n\sum_{m=0}^{n-1}\Lambda(m)\lambda^{n-1-m} + \sum_{n=1}^{n=\bar{M}}\alpha_{-n}\sum_{m=1}^{n}\Lambda(-m)\lambda^{-n-1+m}, \tag{7}$$

where

$$\Lambda(m) = \frac{\lambda_1^m + \lambda_1^{*m}}{2}. \tag{8}$$

$S^{-1}\bar{\partial}S$ are given similarly. The appearance of $S(\lambda,\lambda_1)$ is the most distinguishing feature of the present formalism.

By considering this requirement resulting from equation (1) with $U_m \to U_m^{[N]}$, $V_n \to V_n^{[N]}$, $\Phi \to \Phi^{[N]}$, we can obtain following relation:

$$\sum \lambda^m U_m^{[N]}(\lambda - \sigma) = (\lambda - \sigma) \sum \lambda^m U_m + S^{-1}\partial S(\lambda - \sigma) + \partial(\lambda - \sigma), \qquad (9)$$

and a similar relation for V_n. Explicitly, equation (9) requires that

$$\begin{aligned} U_M^{[N]} &= U_M, \\ U_{M-1}^{[N]} &= U_{M-1} + [U_M, \sigma], \\ U_{M-j}^{[N]} &= U_{M-j} + [U_{M-j+1}, \sigma] + [U_{M-j+2}, \sigma]\sigma + \cdots + [U_M, \sigma]\sigma^{j-1} \\ &\quad - \sum_{m=2}^{j} \alpha_{M-j+m}\{\Lambda(m-1) - \sigma^{m-1}\} \end{aligned} \qquad (10)$$

and

$$\begin{aligned} U_{-\bar{M}}^{[N]} &= \sigma U_{-\bar{M}}\sigma^{-1} + \alpha_{-\bar{M}}\{\Lambda(-1) - \sigma^{-1}\}, \\ U_{-\bar{M}+j}^{[N]} &= \sigma\{U_{-\bar{M}+j} + [U_{-\bar{M}+j-1}, \sigma^{-1}] + \cdots + [U_{-\bar{M}}, \sigma^{-1}]\sigma^{-j+1}\}\sigma^{-1} \\ &\quad + \sum_{m=1}^{j} \alpha_{-\bar{M}+m}\{\Lambda(-j-1+m) - \sigma^{-j-1+m}\}. \end{aligned} \qquad (11)$$

2.4. Darboux invariants of U

The expression for $U_M^{[N]}$ in equation (10) shows that it is invariant under the Darboux transformation. As the Darboux transformation is used to obtain a new solution of field variables like $\psi^{[N]}(z, \bar{z})$ from an old one $\psi(z, \bar{z})$, the invariance of U_M under the Darboux transformation means it should not contain field variables. In the present study, we take it that U_M can have explicit $z, \bar{z}$-dependence such that

$$U_M = f(z, \bar{z})T, \qquad (12)$$

where $T = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $f(z, \bar{z})$ is an arbitrary real *inhomogeneity* function [22]. The inhomogeneity function $f(z, \bar{z})$ should satisfy constraints from the compatibility of the Lax pair, but is not related to the field variables and remains unchanged under the Darboux transformation.

The second equation of equation (10) leads to another Darboux invariant, Tr $(U_M U_{M-1})$. This invariant means that the diagonal part of U_{M-1} is a Darboux invariant such that it cannot contain field variables but could be an arbitrary real function. In this work, we take it to be 0, for simplicity. For more general cases where it is an arbitrary function, see [22] (for cases having constant spectral parameter). Thus, we take

$$U_{M-1} = \begin{pmatrix} 0 & \psi(z, \bar{z}) \\ -\psi(z, \bar{z})^* & 0 \end{pmatrix}. \qquad (13)$$

To find a Darboux invariant which contains U_{M-2}, we first note that

$$\mathrm{Tr}\big(\big(U_{M-1}^{[N]}\big)^2 + 2U_{M-2}^{[N]}U_M^{[N]}\big) = \mathrm{Tr}((U_{M-1})^2 + 2U_{M-2}U_M) + 2\alpha_M \mathrm{Tr}\,[U_M\{\sigma - \Lambda(1)\}], \qquad (14)$$

where we have used equation (10). Equation (14) shows that $\mathrm{Tr}\big(U_{M-1}^2 + 2U_{M-2}U_M\big)$ becomes a Darboux invariant when the structure function $\alpha_M = 0$ for $M \geqslant 2$. The condition $\alpha_M = 0$, $M \geqslant 2$ is a new criterion for integrable equations having variable spectral parameter. In this case, this Darboux invariant restricts the form of U_{M-2} as following:

$$U_{M-2} = -\{2|\psi(z, \bar{z})|^2/f(z, \bar{z}) - 2F(z, \bar{z})\}T + \begin{pmatrix} 0 & \phi(z, \bar{z}) \\ -\phi(z, \bar{z})^* & 0 \end{pmatrix}, \qquad (15)$$

where $\phi(z,\bar{z})$ is another field variable and $F(z,\bar{z})$ is another inhomogeneity function (not a field variable). This criterion, $\alpha_M = 0, M \geqslant 2$, seems too restrictive. But [16] shows that the equations $\bar{M} = 0$ and $M > 1$ in equation (2) can be transformed to equations of $\bar{M} = 0, M = 1$ by introducing a suitable change of the coordinates. This was called as the polynomial case in [16].

Another Darboux invariant which contains U_{-M} is obtained as follows. We first note that P in equation (5) satisfies $\mathrm{Tr}P = 1$, which in turn results in $\mathrm{Tr}\sigma^{-1} = 2\Lambda(-1)$. Applying this relation to the first equation of equation (11) gives a Darboux invariant, $\mathrm{Tr}U^{[N]}_{-\tilde{M}} = \mathrm{Tr}U_{-\tilde{M}}$. For $\alpha_{-\tilde{M}} = 0$, we find another Darboux invariant $\mathrm{Tr}\big(U^{[N]}_{-\tilde{M}}\big)^2 = \mathrm{Tr}(U_{-\tilde{M}})^2$. These Darboux invariants restrict the form of $U_{-\tilde{M}}$ such that

$$U_{-\tilde{M}} = \begin{pmatrix} \mathrm{i}R & Q \\ -Q^* & -\mathrm{i}R \end{pmatrix}, \tag{16}$$

where $R^2 + |Q|^2 = \kappa(z,\bar{z})$, with an inhomogeneity function $\kappa(z,\bar{z})$.

2.5. Darboux invariants of V

2.5.1. Invariants of V_N, V_{N-1}, and $V_{-\tilde{N}}$. The construction of Darboux invariants for V takes similar step as in the case for U. Especially, the Darboux invariance for V_N requires that V_N does not contain field variables. At the same time, the compatibility condition in equation (3) at $k = M + N$ requires that $[U_M, V_N] = 0$. Thus we take

$$V_N = g(z,\bar{z})T, \tag{17}$$

where $g(z,\bar{z})$ is an inhomogeneity function. $g(z,\bar{z})$ will be determined by requiring the compatibility.

Similarly as in Tr $(U_M U_{M-1})$, Tr $(V_N V_{N-1})$ is a Darboux invariant and we take

$$V_{N-1} = \begin{pmatrix} iv_{N-1}(z,\bar{z})/2 & w(z,\bar{z})\psi(z,\bar{z}) \\ -w(z,\bar{z})\psi(z,\bar{z})^* & -iv_{N-1}(z,\bar{z})/2 \end{pmatrix}, \tag{18}$$

where $v_{N-1}(z,\bar{z})$ and $w(z,\bar{z})$ are arbitrary functions. Contrary to the case of U_{M-1}, we take the inhomogeneity function $v_{N-1} \neq 0$ in V_{N-1}.

The invariant containing $V_{-\tilde{N}}$ is obtained similarly as in the case of $U_{-\tilde{M}}$, and in this case becomes $\mathrm{Tr}(V_{-\tilde{N}})$. In the case of $\beta_{-\tilde{N}} = 0$, we find another Darboux invariant, $\mathrm{Tr}(V_{-\tilde{N}})^2$.

2.5.2. Invariant of V_{N-2}. The most distinctive feature of equations with variable spectral parameters arises in the construction of the V_{N-2} term. We first note the following identity that can be derived in a similar way to as in equation (14):

$$\mathrm{Tr}\Big(\big(V^{[N]}_{N-1}\big)^2 + 2V^{[N]}_N V^{[N]}_{N-2}\Big) = \mathrm{Tr}[(V_{N-1})^2 + 2V_N V_{N-2} + 2\beta_N V_N\{\sigma - \Lambda(1)\}]. \tag{19}$$

Recall that the structure functions with $\alpha_M \neq 0, \beta_N \neq 0$ can be transformed to those with $\alpha_M = 0, \beta_N \neq 0$ by the change of coordinates. Thus, proper treatment of equations with $\beta_N \neq 0$ is important. In fact, equations with $\beta_N \neq 0$ are related to interesting physics such as the inhomogeneous Heisenberg ferromagnet.

In the appendix, we calculate Tr $[V_N\{\sigma - \Lambda(1)\}]$, equation (A.7) is for the $M = 1$ case and equation (A.8) is for the $M = 2$ case. For the $M = 1$ case, equations (19) and (A.7) give a Darboux invariant,

$$\mathrm{Tr}\left((V_{N-1})^2 + 2V_N V_{N-2} + \beta_N\, \mathrm{e}^{\int \alpha_1\,\mathrm{d}z} g(z,\bar{z}) \int \frac{V_{N-1}U_0}{g(z,\bar{z})}\, \mathrm{e}^{-\int \alpha_1\,\mathrm{d}z}\,\mathrm{d}z\right). \tag{20}$$

For the $M = 2$ case, we have a Darboux invariant,

$$\mathrm{Tr}\left((V_{N-1})^2 + 2V_N V_{N-2} + \frac{4}{3}\beta_N\,\mathrm{e}^{\int\alpha_1\,\mathrm{d}z} g(z,\bar z)\int \frac{1}{g(z,\bar z)}\left(V_{N-1}U_0 + \frac{1}{2}V_{N-2}U_1\right)\mathrm{e}^{-\int\alpha_1\,\mathrm{d}z}\,\mathrm{d}z\right). \tag{21}$$

These Darboux invariants constrain the matrix elements of V_{N-2}. For the $M = 1$ case, V_{N-2} is given by

$$V_{N-2} = -\left(\frac{2w(z,\bar z)^2|\psi|^2}{g(z,\bar z)} + \beta_N\,\mathrm{e}^{\int\alpha_1\,\mathrm{d}z}\int\frac{2w(z,\bar z)}{g(z,\bar z)}|\psi|^2\,\mathrm{e}^{-\int\alpha_1\,\mathrm{d}z}\,\mathrm{d}z + h_N(z,\bar z)\right)T + \begin{pmatrix} 0 & \psi_3(z,\bar z) \\ -\psi_3(z,\bar z)^* & 0 \end{pmatrix}, \tag{22}$$

where ψ_3 is a functional of field variables and $h_N(z,\bar z)$ is an inhomogeneity function. V_{N-2} for $M = 2$ case can be similarly constructed. The appearance of nonlinear terms including an integration on z is an interesting feature of the present formalism.

2.5.3. Invariants of V_{N-3}. We first find the following identity, which can be derived similarly as in equation (14):

$$\mathrm{Tr}\left(V_{N-1}^{[N]}V_{N-2}^{[N]} + V_N^{[N]}V_{N-3}^{[N]}\right) = \mathrm{Tr}(V_{N-1}V_{N-2} + V_N V_{N-3}) + \beta_N\mathrm{Tr}(V_N\sigma^2 + V_{N-1}\sigma) + \beta_{N-1}\mathrm{Tr}(V_N\sigma). \tag{23}$$

Now the compatibility of the Lax pair (3) at $k = N + M - 1$ (σ_3-component) gives $\alpha_M V_N = \beta_N U_M$. Thus, for equations with $\alpha_M = 0$, we have $\beta_N = 0$. Then, we need to calculate $\mathrm{Tr}(V_N\sigma)$ in equation (23). For $M = 2$, equations (23) and (A.8) give a Darboux invariant of following form:

$$\mathrm{Tr}\left(V_{N-1}V_{N-2} + V_N V_{N-3} + \frac{2}{3}\beta_{N-1}\,\mathrm{e}^{\int\alpha_1\,\mathrm{d}z} g(z,\bar z) \times \int\frac{1}{g(z,\bar z)}\left(V_{N-1}U_0 + \frac{1}{2}V_{N-2}U_1\right)\mathrm{e}^{-\int\alpha_1\,\mathrm{d}z}\,\mathrm{d}z\right). \tag{24}$$

This invariant will be used to construct the diagonal matrix element of V_{N-3} in the following section.

3. Integrable equations with variable spectral parameter

3.1. Case (A): $M = 1$, $\bar M = 0$, $N = 2$, $\bar N = 0$ equations

This case gives the generalized NLSEs with variable spectral parameters. The Darboux invariants on U_i give the following form:

$$U = \lambda U_1 + U_0 = \frac{\mathrm{i}}{2}\lambda f(z,\bar z)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & \psi \\ -\psi^* & 0 \end{pmatrix}. \tag{25}$$

More generally, ψ can be substituted by a functional of ψ. Similarly, the constraints from the Darboux invariance on V give

$$V = \lambda^2 V_2 + \lambda V_1 + V_0 = \lambda^2 g(z,\bar z)T + \lambda\begin{pmatrix} \mathrm{i}v_1/2 & w\psi \\ -w\psi^* & -\mathrm{i}v_1/2 \end{pmatrix} - \left(\frac{2w^2}{g}|\psi|^2 + 2\beta_2\,\mathrm{e}^{\int\alpha_1\,\mathrm{d}z}\int\mathrm{e}^{-\int\alpha_1\mathrm{d}z}\frac{w}{g}|\psi|^2\,\mathrm{d}z + h_2(z,\bar z)\right)T + \begin{pmatrix} 0 & \psi_3 \\ -\psi_3^* & 0 \end{pmatrix}, \tag{26}$$

where g, v_1, h_2 are inhomogeneity functions (invariants under the Darboux transformation) and ψ_3 is a functional of ψ. $w = w(z, \bar{z})$ is introduced such that $w\psi$ is a simplest functional of ψ. The inhomogeneity functions as well as the structure functions α_i, β_i will be determined from the compatibility of Lax pair, equation (3). Note the appearance of a term containing an integral on $|\psi|^2$ in V_0. Equation (22) shows that this term appears only when $\beta_N = \beta_2 \neq 0$.

The compatibility of the Lax pair at the order of $O(\lambda^2)$ gives

$$\partial g - f\beta_2 + 2g\alpha_1 = 0, \qquad fw - g = 0, \tag{27}$$

and we have $w = g/f$. At the order of $O(\lambda^1)$, we have

$$\begin{aligned} &\partial v_1 + v_1\alpha_1 + 2g\alpha_0 - f\beta_1 - \bar{\partial} f = 0,\\ &\psi_3 = \left(\frac{v_1}{f} - \mathrm{i}\frac{g}{f^2}\alpha_1 - \mathrm{i}\frac{\partial g}{f^2} + \mathrm{i}g\frac{\partial f}{f^3}\right)\psi - \mathrm{i}\frac{g}{f^2}\partial\psi. \end{aligned} \tag{28}$$

At the order of $O(\lambda^0)$, we have

$$\begin{aligned} &-\partial h_2 + v_1\alpha_0 - f\beta_0 = 0,\\ &\bar{\partial}\psi - \partial\psi_3 + \mathrm{i}\left(\frac{2w^2}{g}|\psi|^2 + 2\beta_2\,\mathrm{e}^{\int\alpha_1\,\mathrm{d}z}\int \mathrm{e}^{-\int\alpha_1\,\mathrm{d}z}\frac{w}{g}|\psi|^2\,\mathrm{d}z + h_2 + \mathrm{i}w\alpha_0\right)\psi = 0, \end{aligned} \tag{29}$$

such that

$$h_2 = h_2(z, \bar{z}) = \int (v_1\alpha_0 - f\beta_0)\,\mathrm{d}z + \tilde{h}_2(\bar{z}), \tag{30}$$

where $\tilde{h}_2(\bar{z})$ is an arbitrary function. The second equation of equation (29) becomes the equation of motion for ψ.

Now, the existence of $\lambda(z, \bar{z})$, i.e., $\partial\bar{\partial}\lambda(z, \bar{z}) = \bar{\partial}\partial\lambda(z, \bar{z})$, constrains the structure functions α_i, β_i such that

$$\partial\beta_2 + \alpha_1\beta_2 = 0, \qquad \bar{\partial}\alpha_1 = 2\alpha_0\beta_2 + \partial\beta_1, \qquad \alpha_1\beta_0 + \bar{\partial}\alpha_0 = \alpha_0\beta_1 + \partial\beta_0. \tag{31}$$

There are many possibilities for solutions of these constraints. Here, we present some possible interesting cases.

3.1.1. $\beta_2 = 0, \beta_1 = \bar{\partial}M(z, \bar{z}), \alpha_1 = \partial M(z, \bar{z}), \alpha_0 = \beta_0 = 0.$ Equations (27)–(29) give $g = \mathrm{e}^{-2M}$, $f = \partial N\,\mathrm{e}^{-M}$, $v_1 = \bar{\partial}N\,\mathrm{e}^{-M}$ and

$$\psi_3 = \left(\frac{\bar{\partial}N}{\partial N} + \mathrm{i}\frac{\partial^2 N}{(\partial N)^3}\right)\psi - \frac{\mathrm{i}}{(\partial N)^2}\partial\psi, \tag{32}$$

where $M = M(z, \bar{z}), N = N(z, \bar{z})$ are arbitrary functions. (Distinguish these from those in equation (1).) The equation of motion in equation (29) becomes

$$\begin{aligned} &\bar{\partial}\psi + \frac{2\mathrm{i}}{(\partial N)^2}|\psi|^2\psi + \frac{\mathrm{i}}{(\partial N)^2}\partial^2\psi - \left(\frac{\bar{\partial}N}{\partial N} + \frac{3\mathrm{i}\partial^2 N}{(\partial N)^3}\right)\partial\psi\\ &\quad + \left(\frac{\partial^2 N\bar{\partial}N}{(\partial N)^2} - \frac{\partial\bar{\partial}N}{\partial N} + \frac{3\mathrm{i}(\partial^2 N)^2}{(\partial N)^4} - \frac{\mathrm{i}\partial^3 N}{(\partial N)^3} + \mathrm{i}\tilde{h}_2(\bar{z})\right)\psi = 0. \end{aligned} \tag{33}$$

3.1.2. $\beta_2 = \alpha_1 = \beta_1 = 0, \alpha_0 = \partial M(z, \bar{z}), \beta_0 = \bar{\partial}M(z, \bar{z}).$ Equations (27)–(29) give

$$\begin{aligned} &g = g(\bar{z}), \qquad v_1 = \int (\bar{\partial}f - 2g\partial M)\,\mathrm{d}z,\\ &\psi_3 = \left(\mathrm{i}\frac{g\partial f}{f^3} + \frac{1}{f}\int\bar{\partial}f\,\mathrm{d}z - \frac{2gM}{f}\right)\psi - \mathrm{i}\frac{g}{f^2}\partial\psi \end{aligned} \tag{34}$$

and

$$h_2(z,\bar{z}) = -gM^2 - \int \bar{\partial}(fM)\,\mathrm{d}z + M\int \bar{\partial} f\,\mathrm{d}z. \tag{35}$$

The equation of motion, in this case, is (we take $\tilde{h}_2 = 0$)

$$\bar{\partial}\psi + \mathrm{i}\frac{g}{f^2}\partial^2\psi + 2\mathrm{i}\frac{g}{f^2}|\psi|^2\psi - \left(\frac{3\mathrm{i}g\partial f}{f^3} - \frac{2gM}{f} + \frac{\int\bar{\partial} f\,\mathrm{d}z}{f}\right)\partial\psi + \Upsilon\psi = 0, \tag{36}$$

where

$$\Upsilon = -\mathrm{i}\int\bar{\partial}(fM)\,\mathrm{d}z + \mathrm{i}M\int\bar{\partial} f\,\mathrm{d}z - \mathrm{i}\partial\left(\frac{g\partial f}{f^3}\right) - \mathrm{i}gM^2 \\ - \frac{2gM\partial f}{f^2} + \frac{g\partial M}{f} - \partial\left(\frac{\int\bar{\partial} f\,\mathrm{d}z}{f}\right), \tag{37}$$

where $f = f(z,\bar{z})$, $g = g(\bar{z})$.

3.1.3. $\beta_2 = \exp[-M(z,\bar{z})], \alpha_1 = \partial M(z,\bar{z}), \beta_1 = \bar{\partial} M(z,\bar{z}), \alpha_0 = \beta_0 = 0.$ The compatibility of the Lax pair gives (we take $\tilde{h}_2 = 0$)

$$f = \partial N\,\mathrm{e}^{-M}, \qquad g = N\,\mathrm{e}^{-2M}, \qquad v_1 = \bar{\partial} N\,\mathrm{e}^{-M}, \\ \psi_3 = \left(-\frac{\mathrm{i}}{\partial N} + \mathrm{i}\frac{N\partial^2 N}{(\partial N)^3} + \frac{\bar{\partial} N}{\partial N}\right)\psi - \mathrm{i}\frac{N}{(\partial N)^2}\partial\psi, \qquad h_2 = 0, \tag{38}$$

where $N = N(z,\bar{z})$ is an arbitrary function. The equation of motion is

$$\bar{\partial}\psi + \mathrm{i}\frac{N}{(\partial N)^2}\partial^2\psi + 2\mathrm{i}\frac{N}{(\partial N)^2}|\psi|^2\psi + \left(\frac{2\mathrm{i}}{\partial N} - \frac{\bar{\partial} N}{\partial N} - 3\mathrm{i}\frac{N\partial^2 N}{(\partial N)^3}\right)\partial\psi + \Upsilon\psi, \tag{39}$$

where

$$\Upsilon = -\frac{\partial\bar{\partial} N}{\partial N} + \frac{\bar{\partial} N\partial^2 N}{(\partial N)^2} - 2\mathrm{i}\frac{\partial^2 N}{(\partial N)^2} - \mathrm{i}\frac{N\partial^3 N}{(\partial N)^3} + 3\mathrm{i}\frac{N(\partial^2 N)^2}{(\partial N)^4} + 2i\int\frac{|\psi|^2}{\partial N}\,\mathrm{d}z. \tag{40}$$

3.2. *Case (B): $M = 2, \bar{M} = 0, N = 3, \bar{N} = 0$ equation*

We take $\alpha_M = \alpha_2 = \beta_N = \beta_3 = 0$, see discussion in subsection 2.5.3. In this case, we have

$$U = \lambda^2 U_2 + \lambda U_1 + U_0 \\ = \frac{\mathrm{i}}{2}\lambda^2 f(z,\bar{z})\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \lambda\begin{pmatrix}0 & \psi\\ -\psi^* & 0\end{pmatrix} - \frac{2}{f}|\psi|^2 T + \begin{pmatrix}0 & \phi\\ -\phi^* & 0\end{pmatrix}. \tag{41}$$

Here, we take $F = 0$ in equation (15) for simplicity. ψ and ϕ are field variables.

Similarly, the Darboux invariants of V give

$$V = \lambda^3 V_3 + \lambda^2 V_2 + \lambda V_1 + V_0 \\ = \lambda^3 g(z,\bar{z})T + \lambda^2(v_2 T + wU_1) \\ + \lambda\left(-\frac{2w^2}{g}|\psi|^2 T - h_3 T + \begin{pmatrix}0 & \psi_3\\ -\psi_3^* & 0\end{pmatrix}\right) + v_0 T + \begin{pmatrix}0 & \psi_4\\ -\psi_4^* & 0\end{pmatrix}, \tag{42}$$

where h_3, v_2, w are arbitrary real functions. ψ_3 and ψ_4 are functionals of ψ and ϕ. v_0 is determined by the Darboux invariant (24) with $N = 3$ as following. We calculate the Darboux invariant (24) and set it to be an inhomogeneity function $\tilde{v}_0(z,\bar{z})$ such that

$$\begin{aligned}
&\mathrm{Tr}\left(V_2V_1+V_3V_0+\frac{2}{3}\beta_2\,\mathrm{e}^{\int\alpha_1\,\mathrm{d}z}g(z,\bar{z})\int\frac{1}{g(z,\bar{z})}\left(V_2U_0+\frac{1}{2}V_1U_1\right)\mathrm{e}^{-\int\alpha_1\,\mathrm{d}z}\,\mathrm{d}z\right)\\
&\quad=-\frac{g}{2}v_0+\frac{1}{2}v_2\left(\frac{2w^2}{g}|\psi|^2+h_3\right)-w(\psi\psi_3^*+\psi^*\psi_3)\\
&\qquad+\frac{2}{3}\beta_2\,\mathrm{e}^{\int\alpha_1\,\mathrm{d}z}g\int\left(\frac{v_2}{fg}|\psi|^2-w(\psi\phi^*+\psi^*\phi)-\frac{1}{2}(\psi\psi_3^*+\psi_3^*\psi)\right)\mathrm{e}^{-\int\alpha_1\,\mathrm{d}z}\,\mathrm{d}z\\
&\quad=\tilde{v}_0(z,\bar{z}).
\end{aligned}\tag{43}$$

Now, the existence of $\lambda(z,\bar{z})$, i.e., $\partial\bar{\partial}\lambda(z,\bar{z})=\bar{\partial}\partial\lambda(z,\bar{z})$, gives

$$\begin{aligned}
&\partial\beta_3+2\alpha_1\beta_3=0,\qquad \alpha_1\beta_2+3\alpha_0\beta_3+\partial\beta_2=0,\\
&\bar{\partial}\alpha_1-2\alpha_0\beta_2-\partial\beta_1=0,\qquad \alpha_1\beta_0+\bar{\partial}\alpha_0=\alpha_0\beta_1+\partial\beta_0.
\end{aligned}\tag{44}$$

Here, we present one possible solution,

$$\begin{aligned}
&\beta_3=\alpha_2=\alpha_0=0,\qquad \beta_2=\mathrm{e}^{-M(z,\bar{z})},\\
&\beta_1=\bar{\partial}M(z,\bar{z}),\qquad \beta_0=\mathrm{e}^{M(z,\bar{z})},\qquad \alpha_1=\partial M(z,\bar{z}).
\end{aligned}\tag{45}$$

In the following, we take $g(z,\bar{z})=1$, for simplicity. The compatibility of the Lax pair at various orders of λ gives

$$\begin{aligned}
&w=1/f,\qquad f=\frac{3}{2}\mathrm{e}^M\partial M,\qquad \psi_3=\frac{v_2}{f}\psi+\frac{1}{f}\phi,\qquad v_2=\frac{3}{2}\mathrm{e}^M\bar{\partial}M,\\
&\psi_4=\left(\frac{2}{3}\frac{h_1}{\mathrm{e}^M\partial M}+\frac{2}{9}\mathrm{i}\frac{1}{\mathrm{e}^{2M}\partial M}+\frac{4}{9}\mathrm{i}\frac{\partial^2M}{\mathrm{e}^{2M}(\partial M)^3}\right)\psi-\frac{4}{9}\frac{\mathrm{i}}{\mathrm{e}^{2M}(\partial M)^2}\partial\psi\\
&\qquad+\frac{16}{27}\frac{|\psi|^2\psi}{\mathrm{e}^{3M}(\partial M)^3}+\frac{\phi}{\mathrm{e}^{3M}\partial M}\int\mathrm{e}^{3M}(\partial\bar{\partial}M+3\partial M\bar{\partial}M)\,\mathrm{d}z,\\
&h_3=-\mathrm{e}^{2M(z,\bar{z})},\qquad \tilde{v}_0=0,\\
&v_0=-\frac{8}{9}\frac{\phi\psi^*+\phi^*\psi}{\mathrm{e}^{2M}(\partial M)^2}-\frac{4}{3}\frac{\bar{\partial}M\psi\psi^*}{\mathrm{e}^M(\partial M)^2}-\frac{4}{3}\int\frac{\phi\psi^*+\phi^*\psi}{\mathrm{e}^{2M}\partial M}\,\mathrm{d}z,
\end{aligned}\tag{46}$$

and the equation of motion for ψ and ϕ,

$$\begin{aligned}
&\mathrm{i}\bar{\partial}\psi=-\frac{4}{3}\frac{\bar{\partial}M}{\mathrm{e}^M(\partial M)^2}|\psi|^2\psi+\mathrm{i}\frac{\bar{\partial}M}{\partial M}\partial\psi+\left(\frac{2}{3}\mathrm{i}\frac{\partial^2M}{\mathrm{e}^M(\partial M)^2}-\mathrm{e}^{2M}\right)\phi\\
&\qquad+\frac{2}{3}\mathrm{i}\frac{1}{\mathrm{e}^M\partial M}\partial\phi+\mathrm{i}\partial\left(\frac{\bar{\partial}M}{\partial M}\right)\psi-v_0\psi,\\
&\bar{\partial}\phi-\partial\psi_4-\mathrm{i}\frac{2}{f}|\psi|^2\psi_4-\mathrm{i}v_0\phi+\mathrm{e}^M\psi=0.
\end{aligned}\tag{47}$$

3.3. Case (C): $M=\bar{M}=N=\bar{N}=1$ equation

In this case, we have

$$\begin{aligned}
U&=\lambda U_1+U_0+\frac{1}{\lambda}U_{-1}\\
&=\frac{\mathrm{i}}{2}\lambda f(z,\bar{z})\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\begin{pmatrix}0&\psi\\-\psi^*&0\end{pmatrix}+\frac{1}{\lambda}\begin{pmatrix}\mathrm{i}R&Q\\-Q^*&-\mathrm{i}R\end{pmatrix},
\end{aligned}\tag{48}$$

where ψ, R, Q are field variables.

The Darboux invariants for V_i as well as the compatibility of the Lax pair give following form for V:

$$\begin{aligned} V &= \lambda^1 V_1 + V_0 + \frac{1}{\lambda} V_{-1} \\ &= \lambda g(z,\bar{z}) T + v_1 T + v_2 U_0 + v_0 \begin{pmatrix} 0 & Q \\ -Q^* & 0 \end{pmatrix} + \frac{1}{\lambda} v_3 U_{-1}, \end{aligned} \tag{49}$$

where v_0, v_1, v_2, v_3 are arbitrary real functions.

As in the previous examples, the existence of $\lambda(z,\bar{z})$, i.e., $\partial\bar{\partial}\lambda(z,\bar{z}) = \bar{\partial}\partial\lambda(z,\bar{z})$ constrains the structure functions α_i, β_i. Here, we take $\alpha_1 = \partial M, \beta_1 = \bar{\partial} M$ and $a_i = \beta_i = 0, i \neq 1$, such that

$$\lambda = \mu\, \mathrm{e}^{M}, \tag{50}$$

where μ is a complex constant (the hidden spectral parameter) and $M = M(z,\bar{z})$ is a real function.

The compatibility of the Lax pair at various orders of λ gives

$$v_0 = 0, \qquad v_2 = g/f, \qquad v_1 = \text{constant}, \qquad \bar{\partial} f + \beta_1 f = \partial g + \alpha_1 g, \tag{51}$$

which has a solution,

$$f = \partial N\, \mathrm{e}^{-M}, \qquad g = \bar{\partial} N\, \mathrm{e}^{-M}, \tag{52}$$

where $N = N(z,\bar{z})$ is an arbitrary function. Equations of motion for ψ, Q, R are

$$\begin{aligned} &\bar{\partial}\psi - \partial(g\psi/f) + \mathrm{i}(v_3 f - g)Q - \mathrm{i}v_1\psi = 0, \\ &\bar{\partial}Q - \beta_1 Q - \partial(v_3 Q) + \alpha_1 v_3 Q - 2\mathrm{i}v_3 P\psi - \mathrm{i}v_1 Q + 2\mathrm{i}gR\psi/f = 0, \\ &\bar{\partial}R - \beta_1 R - \partial(v_3 R) + \alpha_1 v_3 R + \mathrm{i}(v_3 - g/f)(\psi Q^* - \psi^* Q) = 0. \end{aligned} \tag{53}$$

These equations of motion are consistent with the following Darboux invariant condition:

$$\mathrm{Tr}(V_{-\tilde{N}})^2 = -2v_3^2(|Q|^2 + R^2) = -2v_3^2\kappa(z,\bar{z}), \tag{54}$$

where $\kappa(z,\bar{z})$ $(= -\mathrm{Tr}(U_{-\tilde{M}})^2/2)$ is an inhomogeneity function. In fact,

$$\bar{\partial}\kappa = Q\bar{\partial}Q^* + Q^*\bar{\partial}Q + 2R\bar{\partial}R = v_3\bar{\partial}\kappa + 2\kappa\partial v_3 + 2(\beta_1 - v_3\alpha_1)\kappa, \tag{55}$$

which shows that κ is related only to v_3, α_1, β_1 and not to the field variables ψ, P, Q. As an example, we take $\kappa(z,\bar{z}) = $ constant. In this case, equation (55) gives

$$v_3 = \mathrm{e}^{M}\bar{\partial}\int \mathrm{e}^{-M}\,\mathrm{d}z. \tag{56}$$

4. One-soliton

In this section, we calculate one-soliton solutions of the integrable equations constructed in the previous sections. This can be achieved using the Darboux transformation in section 2.3.

4.1. *One-soliton of case (A)*

The spectral parameter in the case of subsection 3.1.3 is given by

$$\lambda = -\frac{\mathrm{e}^{M(z,\bar{z})}}{\bar{z} - \mu}, \tag{57}$$

where μ is the hidden spectral parameter. Starting from the trivial solution $\psi = 0$, we obtain a solution of the associated linear equation (1),

$$\Phi_1 = \exp\left(-\frac{\mathrm{i}}{2}\frac{N}{\bar{z}-\mu_1}\sigma_3\right)\Phi_0, \tag{58}$$

where $\mu_1 = \mu_r + \mathrm{i}\mu_i$ is a specific value of the hidden spectral parameter and Φ_0 is an arbitrary constant 2-component column matrix. We take $\Phi_0 = \binom{1}{1}$. Define real-valued Δ_r, Δ_i such that

$$-\frac{N}{\bar{z}-\mu_r-\mathrm{i}\mu_i} \equiv \Delta_r + \mathrm{i}\Delta_i. \tag{59}$$

Then, the projector P in equation (5) becomes

$$P = \frac{1}{2}\mathrm{sech}\Delta_i \begin{pmatrix} \mathrm{e}^{-\Delta_i} & \mathrm{e}^{\mathrm{i}\Delta_r} \\ \mathrm{e}^{-\mathrm{i}\Delta_r} & \mathrm{e}^{\Delta_i} \end{pmatrix}. \tag{60}$$

Now, the one-soliton solution $\psi^{[N]}$ is obtained by using equation (10),

$$U_0^{[N]} = \begin{pmatrix} 0 & \psi^{[N]} \\ -\psi^{[N]*} & 0 \end{pmatrix} = [fT, \lambda_1^* + (\lambda_1 - \lambda_1^*)P], \qquad \lambda_1 = -\frac{\mathrm{e}^{M(z,\bar{z})}}{\bar{z}-\mu_1}, \tag{61}$$

such that

$$\psi^{[N]} = -\frac{\partial N}{N}\Delta_i \mathrm{sech}\Delta_i\, \mathrm{e}^{\mathrm{i}\Delta_r}. \tag{62}$$

It was explicitly checked that $\psi^{[N]}$ satisfies the equation of motion in equation (39).

4.2. One-soliton of case (B)

Here, we treat the case $M(z,\bar{z}) = z$. Then, the variable spectral parameter λ determined by equation (45) is

$$\lambda = \tan(\bar{z}-\mu)\,\mathrm{e}^{z}, \tag{63}$$

where μ is the hidden spectral parameter. Starting from the trivial solution $\psi = \phi = 0$, we obtain a solution of equation (1),

$$\Phi_1 = \exp\left(\frac{\mathrm{i}}{4}\tan^2(\bar{z}-\mu_1)\,\mathrm{e}^{3z}\sigma_3\right)\Phi_0, \tag{64}$$

where $\mu_1 = \mu_r + \mathrm{i}\mu_i$. Define Δ_r, Δ_i such that

$$\frac{1}{2}\tan^2(\bar{z}-\mu_1)\,\mathrm{e}^{3z} = \frac{1}{2}\mathrm{e}^{3z}\frac{\{\sin(2\bar{z}-2\mu_r) - \mathrm{i}\sinh 2\mu_i\}^2}{\{\cosh 2\mu_i + \cos(2\bar{z}-2\mu_r)\}^2} \equiv \Delta_r + \mathrm{i}\Delta_i. \tag{65}$$

Then, the projector P becomes that of equation (60) where Δ_r and Δ_i are replaced by equation (65). The one-soliton solution for $\psi^{[N]}$ is similarly obtained as in equation (61),

$$\psi^{[N]} = \frac{3}{2}\mathrm{e}^{2z}\frac{\sinh 2\mu_i}{\cosh 2\mu_i + \cos(2\bar{z}-2\mu_r)}\mathrm{sech}\Delta_i\, \mathrm{e}^{\mathrm{i}\Delta_r}. \tag{66}$$

Now, equations (10) and (41) give

$$\begin{aligned} U_0^{[N]} &= -\frac{4}{3}\mathrm{e}^{-z}|\psi^{[N]}|^2 T + \begin{pmatrix} 0 & \phi^{[N]} \\ -\phi^{[N]*} & 0 \end{pmatrix} \\ &= U_0 + [U_1, \sigma] + [U_2, \sigma]\sigma - \alpha_2\{\Lambda(1) - \sigma\} = [fT, \sigma]\sigma, \end{aligned} \tag{67}$$

such that

$$\phi^{[N]} = \frac{3}{2}\mathrm{e}^{3z+\mathrm{i}\Delta_r}\frac{\sinh 2\mu_i}{\{\cosh 2\mu_i + \cos(2\bar{z}-2\mu_r)\}^2}\mathrm{sech}\Delta_i\{\sin(2\bar{z}-2\mu_r) - \mathrm{i}\sinh 2\mu_i \tanh\Delta_i\}. \tag{68}$$

It was explicitly checked that $\psi^{[N]}$, $\phi^{[N]}$ satisfy the equations of motion in equation (47).

4.3. One-soliton of case (C)

We take $\kappa = 1$ for simplicity. Starting from the trivial solution $\psi = Q = 0$, $R = 1$, we obtain

$$\Phi_1 = \exp\left(\frac{\mathrm{i}}{2}\left(\mu_1 N + \frac{2}{\mu_1}H + v_1\bar{z}\right)\sigma_3\right)\Phi_0, \tag{69}$$

where $H = \int \exp\{-M(z,\bar{z})\}\,\mathrm{d}z$ and $\mu_1 = \mu_r + \mathrm{i}\mu_i$. Define Δ_r, Δ_i such that

$$\mu_1 N + \frac{2}{\mu_1}H = (\mu_r + \mathrm{i}\mu_i)N + 2\frac{\mu_r - i\mu_i}{|\mu_1|^2}H \equiv \Delta_r + \mathrm{i}\Delta_i. \tag{70}$$

Then,

$$P = \frac{1}{2}\mathrm{sech}\Delta_i \begin{pmatrix} \mathrm{e}^{-\Delta_i} & \mathrm{e}^{\mathrm{i}(\Delta_r + v_1\bar{z})} \\ \mathrm{e}^{-\mathrm{i}(\Delta_r + v_1\bar{z})} & \mathrm{e}^{\Delta_i} \end{pmatrix} \tag{71}$$

and

$$\psi^{[N]} = -\mu_i\partial N \mathrm{sech}\Delta_i\, \mathrm{e}^{\mathrm{i}(\Delta_r + v_1\bar{z})}. \tag{72}$$

Equation (11) gives

$$\begin{aligned} U_{-1}^{[N]} &= \begin{pmatrix} \mathrm{i}R^{[N]} & Q^{[N]} \\ -(Q^{[N]})^* & -\mathrm{i}R^{[N]} \end{pmatrix} \\ &= \sigma U_{-1}\sigma^{-1} = \frac{1}{\partial H}(\mu_r - \mathrm{i}\mu_i + 2\mathrm{i}\mu_i P)U_{-1}\frac{\partial H}{|\mu_1|^2}(\mu_r + \mathrm{i}\mu_i - 2\mathrm{i}\mu_i P), \end{aligned} \tag{73}$$

such that

$$\begin{aligned} R^{[N]} &= 1 - 2\frac{\mu_i^2}{|\mu_1|^2}\mathrm{sech}^2\Delta_i, \\ Q^{[N]} &= 2\frac{\mu_i}{|\mu_1|^2}(\mu_r - \mathrm{i}\mu_i \tanh\Delta_i)\mathrm{sech}\Delta_i\, \mathrm{e}^{\mathrm{i}(\Delta_r + v_1\bar{z})}. \end{aligned} \tag{74}$$

It was explicitly checked that $(R^{[N]})^2 + |Q^{[N]}|^2 = 1$ and $\psi^{[N]}$, $R^{[N]}$, $Q^{[N]}$ satisfy the equations of motion in equation (53).

5. Discussion

In this paper, we study the Darboux transformation of integrable equations with variable spectral parameters. We construct Darboux invariants of U and V, which give constraints on the matrix elements of U and V. It gives important criteria in constructing integrable equations. Especially, we need $\alpha_M = 0$ for $M \geqslant 2$. For $\beta_N \neq 0$ or $\beta_{N-1} \neq 0$ cases, specific forms of V_{N-1}, V_{N-2}, V_{N-3} are obtained using the Darboux transformation property.

Burtsev *et al* [16] constructed a simple type of equation with variable spectral parameters using ansatz on forms of U_i and V_i and applying the compatibility. But this method does not give systematic understanding of the appearance of nonlinear terms in equation (22) or (24), or various inhomogeneity functions resulting from the Darboux invariants. Our formalism *systematically* determines U_i or V_i of integrable equations , which is effective especially for equations of complex nature.

In the present work, we have not studied the most general form of Darboux invariants and many inhomogeneity functions are set to be zero or constant, even though they can be explicit functions of z and $\bar{z}$. This generalization remains an interesting work.

The Darboux covariance method was used to construct (2 + 1)-dimensional integrable equations [21] and multi-variables equations, where U and V span the Hermitian symmetric spaces [23]. Present formalism can provide the construction of integrable equations of these kinds of equations with variable spectral parameters.

Acknowledgment

This work was supported by Korea Research Foundation Grant (KRF-2003-070-C00011).

Appendix

Firstly, we obtain a differential equation for σ such that

$$\partial\sigma - \alpha_{-\tilde{M}}\Lambda(-\tilde{M})(1-\delta_{\tilde{M},1}) - \sum_{m=-\tilde{M}+1}^{M}\alpha_m\sigma^m + 2\sum_{m=2}^{M}\alpha_m\sum_{n=0}^{m-2}\sigma^{m-n}\Lambda(n)$$
$$- \sum_{m=0}^{M}[U_m,\sigma]\sigma^m + \sigma\sum_{m=1}^{\tilde{M}}[U_{-m},\sigma^{-1}]\sigma^{-m+1} = 0, \qquad \text{(A.1)}$$

where we use the $O(\lambda^0)$ term of equation (9), the last equation of equation (10), (7) and (2). The formal solution of equation (A.1) for the $M = 1$, $\tilde{M} = 0$ case becomes

$$\sigma = \mathrm{e}^{\int\alpha_1\,\mathrm{d}z}\left(\int([U_0,\sigma] + [U_1,\sigma]\sigma + \alpha_0)\,\mathrm{e}^{-\int\alpha_1\,\mathrm{d}z}\,\mathrm{d}z + c\right), \qquad \text{(A.2)}$$

while for the $M = 2$, $\tilde{M} = 0$ case,

$$\sigma = \mathrm{e}^{\int\alpha_1\,\mathrm{d}z}\left(\int([U_0,\sigma] + [U_1,\sigma]\sigma + [U_2,\sigma]\sigma^2 - \alpha_2\sigma^2 + \alpha_0)\,\mathrm{e}^{-\int\alpha_1\,\mathrm{d}z}\,\mathrm{d}z + c\right). \qquad \text{(A.3)}$$

Now, we calculate $\mathrm{Tr}\,[V_N\{\sigma - \Lambda(1)\}]$ for the case of $\bar{M} = 0$, $M = 1$. We first note that $\mathrm{Tr}\,[V_N\Lambda(1)] = \Lambda(1)g(z,\bar{z})\mathrm{Tr}T = 0$. Now, the following identities are required:

$$\mathrm{Tr}(T[U_1,\sigma]\sigma) = \tfrac{1}{2}\mathrm{Tr}\,([\sigma,T][U_1,\sigma]), \qquad \text{(A.4)}$$

where we used $[U_1,T] = f(z,\bar{z})[T,T] = 0$, and

$$\mathrm{Tr}(T[U_0,\sigma]) = \mathrm{Tr}([\sigma,T]U_0). \qquad \text{(A.5)}$$

By using these results, we obtain

$$\mathrm{Tr}(T[U_1,\sigma]\sigma + T[U_0,\sigma]) = \frac{1}{2g(z,\bar{z})}\mathrm{Tr}\big([\sigma,V_N]\big(U_0^{[N]} + U_0\big)\big)$$
$$= \frac{1}{2g(z,\bar{z})}\mathrm{Tr}\big(V_{N-1}U_0 - V_{N-1}^{[N]}U_0^{[N]}\big), \qquad \text{(A.6)}$$

where we have used equation (10) and $\mathrm{Tr}V_{N-1}U_0^{[N]} = \mathrm{Tr}V_{N-1}^{[N]}U_0$, see equations (13) and (18). By collecting all these results and using equation (A.2), we finally obtain

$$\mathrm{Tr}\,[V_N\{\sigma - \Lambda(1)\}]$$
$$= \mathrm{e}^{\int\alpha_1\,\mathrm{d}z}g(z,\bar{z})\int\frac{1}{2g(z,\bar{z})}\mathrm{Tr}\big(V_{N-1}U_0 - V_{N-1}^{(N)}U_0^{(N)}\big)\,\mathrm{e}^{-\int\alpha_1\,\mathrm{d}z}\,\mathrm{d}z. \qquad \text{(A.7)}$$

Now, we calculate $\mathrm{Tr}\,(V_N\{\sigma - \Lambda(1)\})$ for the case of $\bar{M} = 0$, $M = 2$ and $\alpha_2 = 0$. By using equation (A.3), a similar procedure as in $\bar{M} = 0$, $M = 1$ case gives

$$\mathrm{Tr}\,[V_N\{\sigma - \Lambda(1)\}] = \frac{2}{3}\,\mathrm{e}^{\int\alpha_1\,\mathrm{d}z}g(z,\bar{z})$$
$$\times\int\frac{1}{g(z,\bar{z})}\mathrm{Tr}\left(V_{N-1}U_0 - V_{N-1}^{(N)}U_0^{(N)} + \frac{1}{2}V_{N-2}U_1 - \frac{1}{2}V_{N-2}^{(N)}U_1^{(N)}\right)\mathrm{e}^{-\int\alpha_1\,\mathrm{d}z}\,\mathrm{d}z. \qquad \text{(A.8)}$$

Here we have used equations (11) and their corresponding equations for V_m, as well as the fact that $V_N = g(z,\bar{z})T$ and $[U_2,V_{N-2}] + [U_1,V_{N-1}] + [U_0,V_N] = 0$ (see equation (3)).

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